Welcome to Math 20B: Calculus for Science & Engineering 2

Your Narrator: Prof. Rayan Saab

Course Webpage: math.ucsd.edu/~rsaab → teaching → this course

Contains/will contain: Syllabus
Exam Schedule
HW
Office Hours
TA Information

Book ← Ryagawski (Early Transcendentals) 2nd edition (sorry about price)

Grading Scheme: HW, 2MTs, Final

20%, 20%, 20%, 40% (Final)

or 20% (HW), 20% (Highest MT), 60% (Final)

(Whichever is higher)

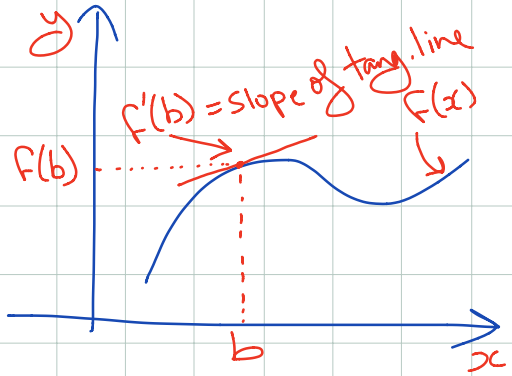
There is a "Math 20B student supplement" that we will cover parts of. Please download it off of course webpage.

Review

Calculus can be divided into 2 parts

For the moment, let's work with continuous functions

Differential Calculus

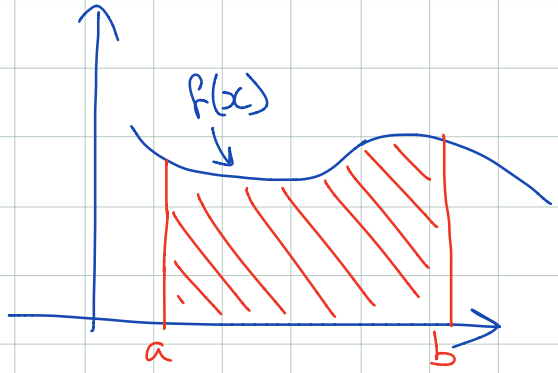


$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

definition

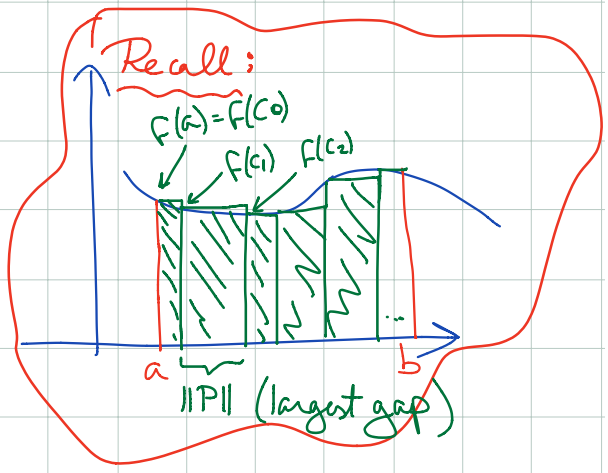
$f'(x)$ is the rate of change of the function at x

Integral Calculus



$$\int_a^b f(x)\,dx := \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

definition (Riemann Sum)



$\int_a^b f(x)\,dx$ is the signed area under the graph of $f(x)$

The two (rates of change & area under a curve) seem unrelated, except for the Fundamental Theorem of Calculus

Assume that $f(x)$ is cont. on $[a,b]$ & let $F(x)$ be an antiderivative of $f(x)$ on $[a,b]$ (i.e. $F'(x) = f(x)$)

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

or $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In words: If you "sum up" the rate of change of $F(x)$ between a & b (the left hand side) you get $F(b) - F(a)$

Example: Let $f(x) = x^2$

Find the area under the curve $y = f(x)$ between $x = 1$ & $x = 3$

$$\text{Area} = \int_1^3 x^2\,dx$$

$F(x) = \frac{x^3}{3}$ is an anti deriv. of x^2 (why?)

$$\Rightarrow \text{Area} = \int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{1}{3}3^3 - \frac{1}{3}1^3$$

$$= 9 - \tfrac{1}{3}$$

Example:

Let $V(t) = 60 + 10\cos t$

Find $\int_a^b V(t)\,dt$

In fact, let's give this "some meaning"

Example: A car travels along a straight highway with velocity $v(t) = 60 + 10\cos(t)$ km/hr

Find the car's displacement between $t=0$ and $t=1$ hr.

Solution: Recall that velocity is the rate of change of position. Let's call the position $F(t)$

Then $F'(t) = v(t)$. The question is asking us for $F(1) - F(0)$ = displacement along the st. line

$F(1)$: final pos'n; $F(0)$: initial position

By FTOC $F(1) - F(0) = \int_0^1 F'(t)\,dt$

$= \int_0^1 60 + 10\cos t \, dt$

$= 60t + 10\sin(t) \Big|_0^1 = 60 + 10\sin 1$

antiderivative of 60; antider. of $10\cos t$.

The FTOC, part (II)

If $f(x)$ is cont. then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

("Area function")

So we have $\frac{d}{dx} A(x) = f(x)$

i.e. $$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Example: $$\frac{d}{dx}\int_a^x e^{t^3} t^7 \cos t\,dt = e^{x^3} x^7 \cos x$$

↑ by FTOC part II

Example: $$\frac{d}{dx}\int_x^a t^3 \ln t\,dt = -\frac{d}{dx}\int_a^x t^3 \ln t\,dt$$

(why?)

$$= -x^3 \ln x$$

Example: $\frac{d}{dx}\int_0^{x^2} te^t\,dt$ ← problem

what we know how to work with

defining $A(x) := \int_0^{x} te^t\,dt$

we are asked to find $\frac{dA(x^2)}{dx}$

Chain rule! $\frac{d}{dx}A(x^2) = 2x \cdot \frac{d}{dx}A(x)$

$= 2x \frac{d}{dx}\int_0^x te^t\,dt = 2x \cdot x e^x$

FTOC, II

In general $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}A(g(x))$

$= A'(g(x)) \cdot g'(x)$

So: $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$

Some examples (Sec. 5.5)

The integral of velocity:

If $s(t)$ is the pos'n at time t of an object in linear motion (like a car on a straight road) then $v(t) = s'(t)$ is it's velocity

and $$\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} s'(t)\,dt = s(t_2) - s(t_1)$$

Displacement (net change in pos'n)

On the other hand: $$\int_{t_1}^{t_2} |v(t)|\,dt = \text{Distance traveled during } [t_1, t_2]$$

Remember: -ve velocity means "moving backwards"

Exercise: Let $s(t) = 1 + 2\cos \pi t$

- Find the distance traveled between $t_1 = 0$ & $t_2 = 1$
- Find the displacement.
- Are they different? why?

Total Vs Marginal Cost

$C(x)$: cost of producing x units

$C'(x)$: marginal cost

$\int_a^b C'(x) = C(b) - C(a)$ = cost of increasing production from a to b units

Example: The marginal cost of producing x paper clips (in units of 1000) is $1000 + 3x^2 - 50x$ dollars

What is the cost of increasing production from 1000 to 2000?

Soln: We need $C(2) - C(1)$

$$= \int_1^2 C'(x)\,dx = \int_1^2 1000 + 3x^2 - 50x \; dx$$

$$= 1000x + x^3 - 25x^2 \Big|_1^2$$

$$= 2\times10^3 + 2^3 - 25\times2^2 - 1\times10^3 - 1^3 + 25\times1^2$$

$$= \$932$$

— x —

Substitution Method (5.6)

Integration can be a little trickier than differentiation, but there important general techniques that are very helpful.

"Substitution method = Chain rule in reverse"

Ex: If $F(x) = \sin x^2$

then $F'(x) = 2x\cos(x^2)$ [by the Chain rule]

So $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

Recall: (chain rule) If $F'(u) = f(u)$

$$\frac{d}{dx}F(u(x)) = F'(u(x))\,u'(x)$$

$$= f(u(x))\,u'(x)$$

$$\Rightarrow \int f(u(x))\,u'(x)\,dx = F(u(x)) + C$$

Subst. method.

Example: $\frac{d}{dx}\sin(x^3) = 3x^2\cos x^3$

(by chain rule)

So $\int \underbrace{3x^2}_{\text{deriv. of inside function}} \cos(\underbrace{x^3}_{\text{inside function}})\,dx = \sin(x^3) + C$

Substitution using differentials

If $u(x)$ is a function of x,

then $\frac{du}{dx} = u'(x)$

"multiply both sides by dx"

$\Rightarrow du = u'(x)\,dx$

So now $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

$= F(u) + C$

Back to our example:

evaluate $\int 3x^2 \cos(x^3)\,dx$

Soln: Let $u = x^3$ ← step 1

then $du = 3x^2\,dx$ ← step 2

and $\int 3x^2 \cos(x^3)\,dx = \int \cos(\underbrace{x^3}_{u})\,\underbrace{3x^2\,dx}_{du}$

step 3 →

step 4 (solve)

$$= \int \cos u \, du = \sin u + C$$

$$= \sin(x^3) + C$$

↑ step 5 (sub. x back in)

Example: Find

$$\int (x^2+3)^4 x \, dx$$

Soln: Let $u = x^2 + 3$ — (1)

then $du = 2x \, dx$ — (2)

$$\Rightarrow \int (x^2+3)^4 x \, dx = \frac{1}{2} \int u^4 \cdot du \quad (3)$$

$2x\,dx = du$

$\Rightarrow x\,dx = \frac{du}{2}$

$$= \frac{1}{2} \frac{u^5}{5} + C \quad (4)$$

$$= \frac{1}{2} \frac{(x^2+3)^5}{5} + C \quad (5)$$

Example: Evaluate $\int \tan\theta \, d\theta$

Soln $\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta} \, d\theta$

Let $u = \cos\theta$

$\Rightarrow du = -\sin\theta \, d\theta$

$\Rightarrow \int \tan\theta \, d\theta = -\int \frac{1}{\cos\theta}(-\sin\theta) \, d\theta = -\int \frac{1}{u} \, du$

$= -\ln|u| + C$

$= -\ln|\cos\theta| + C$

Example:

Evaluate $\int x\sqrt{2x+1} \, dx$

Soln: "guess" $u = 2x+1$

$du = 2 \, dx$

$\Rightarrow \int x\sqrt{2x+1} \, dx = \frac{1}{2}\int \left(\frac{u-1}{2}\right)\sqrt{u} \, du$

$x \to \frac{u-1}{2}$, $\sqrt{2x+1} \to \sqrt{u}$, $dx \to \frac{du}{2}$

$u = 2x+1$

$\Rightarrow x = \frac{u-1}{2}$

$$= \frac{1}{2}\int \frac{u^{3/2}}{2}\,du - \frac{1}{4}\int u^{1/2}\,du$$

$$= \left(\frac{1}{4}\right)\left(\frac{2}{5}\right)u^{5/2} - \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)u^{3/2} + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

Check $\frac{d}{dx}\left[\frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C\right]$

$$= \frac{1}{4}\cdot 2\cdot(2x+1)^{3/2} - \frac{1}{4}\cdot 2\cdot(2x+1)^{1/2} + 0$$

$$= \frac{1}{2}(2x+1)^{1/2}\left[2x+1-1\right] = x\sqrt{2x+1}$$ ✓

— x —

Change of Variables formula for Definite integrals

Suppose we want $\int_a^b f(u(x))\,u'(x)\,dx$ and suppose that $F(x)$ is an antiderivative of $f(x)$

Then $F(u(x))$ is an antiderivative of $f(u(x))\,u'(x)$
(bec $F(u(x))' = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$)

So by the FTOC

$$\int_a^b F(u(x))\,u'(x)\,dx = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} F(u)\,du$$

Example: Evaluate $\int_0^2 x^3\sqrt{x^4+1}\,dx$

Soln: Let $u = x^4+1$

so $du = 4x^3\,dx \Rightarrow \frac{1}{4}du = x^3\,dx$

$$\Rightarrow \int_0^2 x^3\sqrt{x^4+1}\,dx = \frac{1}{4}\int_{u(0)}^{u(2)} u^{1/2}\,du = \frac{1}{4}\int_1^{17} u^{1/2}\,du$$

$$= \frac{1}{6}u^{3/2}\Big|_1^{17} = \frac{1}{6}\left[17^{3/2} - 1\right]$$